RING THEORY-I

OUTLINES

INTRODUCTION
DEFINITIONS
BASIC PROPERTIES
ZERO DIVISORS & NO ZERO DIVISORS
ADDITIONAL INFORMATION
CONCLUSION

PRESENTED BY

S.Lt / Dr.D.CH.PAPARAO. M.Sc, Ph.D, PGDCA

HEAD, Dept. of Mathematics

NCC Officer NAVY

COORDINATOR, IGNOU STUDY CENTRE.

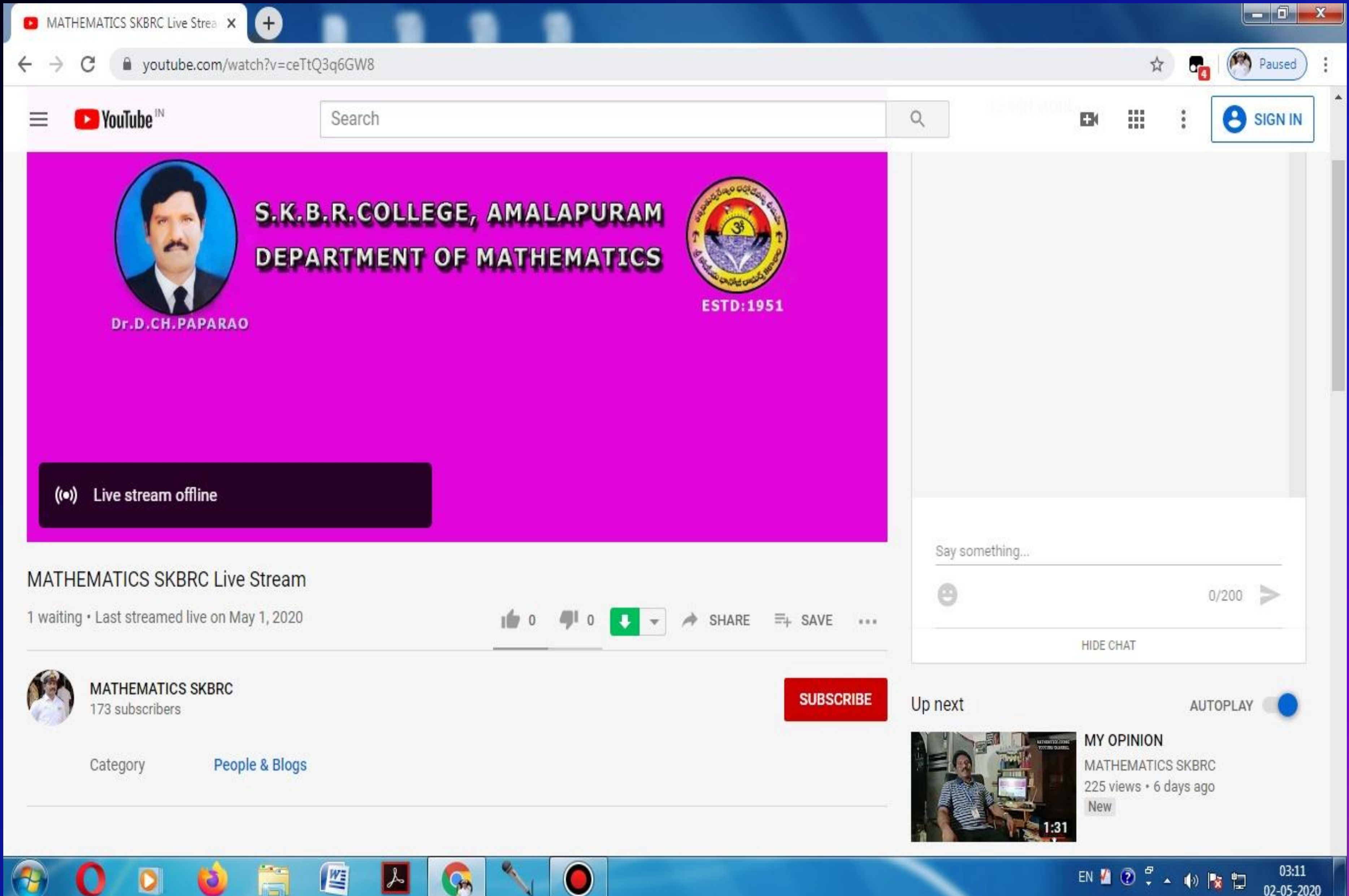
S.K.B.R.COLLEGE, AMALAPURAM

East Godavari Dist. AP-533201

Mail ID: dchp876@gmail.com

YouTube Channel: “MATHEMATICS SKBRC”

My YouTube channel: "MATHEMATICS SKBRC"



INTRODUCTION

In mathematics, a ring is one of the fundamental algebraic structures used in abstract algebra. It consists of a set equipped with two binary operations that generalize the arithmetic operations of addition and multiplication.

Through this generalization, theorems from arithmetic are extended to non-numerical objects such as polynomials, series, matrices and functions.

We have earlier worked with algebraic systems, namely semigroups, monoids and groups, where there is only one binary operation in each.

Now we initiate the study of algebraic systems having two binary operations. For example consider the real number system where we have both addition as well as multiplication.

Likewise the set of all nxn matrices and complex numbers and the set of integers.

DEFINITION OF A RING

An algebraic structure (R,+, .) is called a ring if R is a nonempty set and + a and . are two binary operations satisfying the following axioms

1. **(R, +) is an abelian group**
2. **(R, .) is a semi group and**
3. **Distributive laws are hold**

i.e., $a \cdot (b + c) = a \cdot b + a \cdot c$

$(a + b) \cdot c = a \cdot c + b \cdot c$

for all a, b, c in R.

EXAMPLES FOR RING

1. Any abelian group $(G,+)$ is a ring by defining $a \, . \, b = 0$ for all a and b in G. Therefore $(G, +, .)$ is a ring.

2. The set $(Z, +, .)$ of integers is a ring.
3. The set $(Q, +, .)$ of rational numbers is a ring.
4. The set $(R, +, .)$ of real numbers is a ring.
5. The set $(C, +, .)$ of complex numbers is a ring.

6. The set of all 2x2 matrices over R is a ring w.r.t matrix multiplication.

EXAMPLE FOR RING

Let $\left(R_1,+,\bullet\right),\left(R_2,+,\bullet\right)\cdots\cdots\left(R_n,+,\bullet\right)$ be rings and

$R = R_1 \times R_2 \times \cdots\cdots \times R_n$ and $a, b \in R$,

where $a = \left(a_1, a_2 \cdots\cdots a_n\right)$, $b = \left(b_1, b_2 \cdots\cdots b_n\right)$

$\Rightarrow a + b = \left(a_1 + b_1,\ a_2 + b_2, \cdots\cdots\ a_n + b_n\right)$ and

$\Rightarrow a \cdot b = \left(a_1 \cdot b_1,\ a_2 \cdot b_2, \cdots\cdots a_n \cdot b_n\right)$

then $\left(R,+,\bullet\right)$ is a ring.

Note : The operations + and • defined on R are called the coordinate-wise operations.

EXAMPLE FOR RING

Let $(R, +, \bullet)$ be a ring and X, any non-empty set and R^X be the set of all mappings of X into R.

For any f, $g \in R^X$, we define $f + g : X \to R$ as $(f+g)(x) = f(x) + g(x)$ and $f \cdot g : X \to R$ as $(f \cdot g)(x) = f(x) \cdot g(x)\ \forall x \in X$

Then $(R^X, +, \cdot)$ is a ring.

<u>Note</u> : The operations + and • defined on R^X are called the pointwise addition and pointwise multiplication.

NULL RING OR ZERO RING

Let R={0}. If the addition + and the multiplication . are defined in R as

0+0=0 and 0.0=0 then (R,+,.) is a ring. This ring is called the zero ring.

COMMUTATIVE RING

A ring (R,+,.) is said to be a commutative ring if a.b = b.a for all a, b in R.

Ex: The rings Z, Q, R and C are commutative rings.

Non Commutative Ring

A ring R is said to be a non-commutative ring if R is not commutative.

EX : The set of all 2x2 matrices $M_2(C)$ over the field of complex numbers is a ring, but not a commutative ring.

$$\text{Take } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}, \ BA = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\therefore AB \neq BA$$

Basic properties of a Ring

Theorem : Let R be a ring and a,b,c $\in$ R. Then

$$1)\ \mathbf{a}\cdot 0 = 0\cdot \mathbf{a} = 0 \quad 2)\ \mathbf{a}(-\mathbf{b}) = (-\mathbf{a})\mathbf{b} = -(\mathbf{ab})$$

$$3)\ (-\mathbf{a})(-\mathbf{b}) = \mathbf{ab} \quad 4)\ \mathbf{a}(\mathbf{b}-\mathbf{c}) = \mathbf{ab}-\mathbf{ac} \ \text{ and}$$

$$5)\ (\mathbf{b}-\mathbf{c})\mathbf{a} = \mathbf{ba}-\mathbf{ca}$$

Proof 1 : Let R be a ring and a, 0 $\in$ R

$$\because 0+0=0 \Rightarrow \mathbf{a}(0+0) = \mathbf{a}0 \quad \Rightarrow \mathbf{a}(0)+\mathbf{a}(0) = \mathbf{a}(0)+0$$

$$\Rightarrow \mathbf{a}(0) = 0 \quad (\text{by using left cancelaion law})$$

Similarly we prove that $\mathbf{0}(\mathbf{a}) = 0$

$$\therefore \mathbf{a}(0) = 0(\mathbf{a}) = 0$$

Proof – 2. **We prove that $a(-b) = (-a)(b) = -(ab)$**

$$\because b - b = 0 \quad \Rightarrow a\left[b + (-b)\right] = a(0)$$

$$\Rightarrow a(b) + a(-b) = 0 \quad \Rightarrow a(-b) = -(ab)$$

Similarly we prove that $(-a)(b) = -(ab)$

$$\therefore\ a(-b) = (-a)(b) = -(ab)$$

Proof – 3. **Let R be a ring and $a, b \in R$**

$$LHS = (-a)(-b) = -\left[a(-b)\right] = -\left[-(ab)\right] = ab = RHS$$

$$\therefore (-a)(-b) = ab$$

$$\text{Proof}-4: \underline{\text{We show that } a(b-c)=ab-ac}$$

$$\text{LHS}=a(b-c)=a\left[b+(-c)\right]=a(b)+a(-c)$$

$$=ab+\left[-(ac)\right]=ab-ac=\text{RHS}$$

$$\therefore a(b-c)=ab-ac$$

$$\text{Proof}-5: \underline{\text{We show that } (b-c)a=ba-ca}$$

$$\text{LHS}=(b-c)a=\left[b+(-c)\right]a=ba+(-c)a$$

$$=ba+\left[-(ca)\right]=ba-ca=\text{RHS}$$

$$\therefore (b-c)a=ba-ca$$

BOOLEAN RING

Let R be a ring. R is said to be a boolean ring if $a^2 = a \ \forall a \in R$ (all elemens are idempotent).

Ex: Consider a ring $(Z_2, \oplus_2, \odot_2)$, where $Z_2 = \{0,1\}$

Clearly Z_2 is a boolean ring $\because 0^2 = 0,\ 1^2 = 1$

Idempotent Element:

Let R be a ring and $a \in R$. 'a' is said to be an idempotent element of R if $a^2 = a \ \forall a \in R$.

Note: In a ring, 0 is only idempotent element.

Nilpotent Element

Let R be a ring and $a \in R$. 'a' is said to be a nilpotent element of R if there exists +ve integer n such that $a^n = 0$.

Example: Consider a ring $(Z_9, \oplus, \odot)$ w.r.t addition and multiplication modulo 9.

Clearly $\overline{3}, \overline{6}$ are nilpotent elements in Z_9.

$$\because \left(\overline{3}\right)^2 = \overline{9} = \overline{0}, \quad \left(\overline{6}\right)^2 = \overline{36} = \overline{0}.$$

<u>Theorem</u>: Let R be a boolean ring and $a,b \in R$

Then i) $a + a = 0$ ii) $a + b = 0 \Rightarrow a = b$ iii) $ab = ba$

Proof i) $\because$ **R be a boolean ring** $\Rightarrow a^2 = a \ \forall a \in R$

Let $a \in R \Rightarrow a + a \in R \Rightarrow (a + a)^2 = a + a$

$\Rightarrow (a + a)(a + a) = a + a \Rightarrow a(a + a) + a(a + a) = a + a$

$\Rightarrow (a^2 + a^2) + (a^2 + a^2) = a + a \quad$ (**Using LCL**)

$\Rightarrow (a + a) + (a + a) = (a + a) + 0 \quad \Rightarrow a + a = 0$

ii) $\cdot \quad \because a + a = 0$ **for** $a \in R$**. Let** $a,b \in R$

$a + b = 0 \quad \Rightarrow a + b = a + a \quad \Rightarrow b = a \quad$ (**LCL**)

$\therefore \quad a + b = 0 \quad \Rightarrow a = b$

iii) Let $a,b \in R \Rightarrow a+b \in R \Rightarrow (a+b)^2 = a+b$

$\Rightarrow (a+b)(a+b) = a+b \Rightarrow a(a+b)+b(a+b) = a+b$

$\Rightarrow (a^2+ab)+(ba+b^2) = a+b \quad \because a^2 = a \ \forall a \in R$

$\Rightarrow (a+ab)+(ba+b) = a+b \quad \because x \in R \Rightarrow x+0 = x$

$\Rightarrow (a+b)+(ab+ba) = (a+b)+0 \quad (\text{Using LCL})$

$\Rightarrow ab+ba = 0 \Rightarrow ab = ba \quad (\therefore a+b = 0 \Rightarrow a = b)$

Note: Every boolean ring is commutative. But converse need not be true. Ex: the ring of integers $(Z,+,\cdot)$ is a commutative ring but not a boolean ring.

Here $0^2 = 0,\ 1^2 = 1,\ 2^2 \neq 2, 3^2 \neq 3 \cdots\cdots\cdots$ etc.

ZERO DIVISORS

Let R be a ring and $a \neq 0, b \neq 0 \in R$. a and b are said to be zero divisors of R if $ab = 0$.

$$\therefore \overline{\underline{a \neq 0, b \neq 0 \;\Rightarrow ab = 0}}$$

here 'a' is the left zero divisor and 'b' is the right zero divisor.

<u>Example – I</u> : We know that $\left(Z_6, \oplus, \odot\right)$ is a ring w.r.t addition and multiplication modulo 6.

Take $\bar{2}, \bar{3} \in Z_6 \Rightarrow \bar{2} \odot \bar{3} = \bar{6} = \bar{0}$

$\therefore \bar{2} \neq \bar{0}, \overline{3} \neq \bar{0} \Rightarrow \bar{2} \odot \bar{3} = \bar{0} \;\Rightarrow\; Z_6$ **has zero divisors.**

NO ZERO DIVISORS (OR) WITHOUT ZERO DIVISORS

Let R be a ring and $a, b \in R$. R is said to have no zero divisors if $\underline{ab = 0 \Rightarrow a = 0 \text{ or } b = 0}$

(OR)

$\underline{a \neq 0, b \neq 0 \Rightarrow ab \neq 0}$ for $a, b \in R$

Note: If $a \neq 0$ and $ab = 0$ then we get $b = 0$

<u>Example</u>: We know that $(Z, +, \cdot)$ is a ring

Take $2, 3 \in Z \Rightarrow 2 \cdot 3 = 6 \neq 0$

Clearly $2 \neq 0,\ 3 \neq 0 \Rightarrow 2 \cdot 3 \neq 0$

$\therefore$ The ring Z has no zero divisors.

INTEGRAL DOMAIN

Let R be a ring and $a, b \in R$. R is said to be an integral domain if

1. **R is commutative i.e., $ab = ba \ \forall a, b \in R$**
2. **R has a unity element**
3. **R has no zero divisors**

i.e., $ab = 0 \Rightarrow a = 0$ or $b = 0$

Ex – I : We know that $(Z, +, \cdot)$ is a commutative ring and having unity element $e = 1$ and has no zero divisors.

$\therefore$ Z is an integral domain.

Examples : The rings Q, R and C are integral domains.

CANCELLATION LAWS

Let R be a ring and $a, b, c \in R$.

i) $a \neq o,\ ab = ac \Rightarrow b = c$

This is the left cancellation law

ii) $a \neq o,\ ba = ca \Rightarrow b = c$

This is the right cancellation law

<u>Theorem</u> : Prove that a ring R has no zero divisors iff cancellation laws are hold.

Proof: Suppose R has no zero divisors,

by definition $ab = 0 \Rightarrow a = 0$ **or** $b = 0$

Let $a \neq o,\ b,\ c \in R$ **and** $ab = ac \Rightarrow ab - ac = 0$

$\Rightarrow a(b-c) = 0 \quad \Rightarrow b - c = 0 \quad \Rightarrow b = c \qquad \because a \neq o$

$\therefore a \neq o,\ ab = ac \Rightarrow b = c$ **(i.e., LCL holds)**

Similarly we prove that $a \neq o,\ ba = ca \Rightarrow b = c$

(i.e., RCL holds)

Conversly suppose that canecllation laws are hold

Now we prove that R has no zero divisors

If possible suppose that R has zero divisors

i.e., $\mathbf{a} \neq 0, \mathbf{b} \neq 0 \Rightarrow \mathbf{ab} = 0 \qquad \rightarrow (1)$

For $\mathbf{a} \neq 0$ **&** $\mathbf{ab} = 0 \Rightarrow \mathbf{ab} = \mathbf{a}0 \quad \Rightarrow \mathbf{b} = 0$

Also $\mathbf{b} \neq 0$ **&** $\mathbf{ab} = 0 \Rightarrow \mathbf{ab} = 0\mathbf{b} \quad \Rightarrow \mathbf{a} = 0$

which are not true. $\left[\textbf{from}(1)\right]$ **Hence our supposition R has zero divisors is false.**

$\therefore$ **R has no zero divisors**

Hence proved

The remaining part to be continued in RING THEORY-II video.

ADDITIONAL INFORMATION

The following web links are very useful on internet that will make you smarter, and help to learn new skills.

http://math.bu.edu/people/svh/RingTheoryMathcamp.pdf

https://www.math.uci.edu/~ruiw10/pdf/alg2.pdf

https://en.wikipedia.org/wiki/Ring_theory

https://www.youtube.com/watch?v=DG_hXMdSd1c

https://www.youtube.com/watch?v=_RTHvweHlhE

https://www.youtube.com/watch?v=nVXnpGkILSs

CONCLUSION

I have added definitions, basic concepts of ring, examples and theorems in the chapter ring theory in brief and short.

I hope that utilizing all of these methods through PPT Slides helps to engage students with different types of learning styles.

I have concluded that PPT presentation is very useful in establishing objectives, illustrating concrete examples.

S.K.B.R.COLLEGE, AMALAPURAM